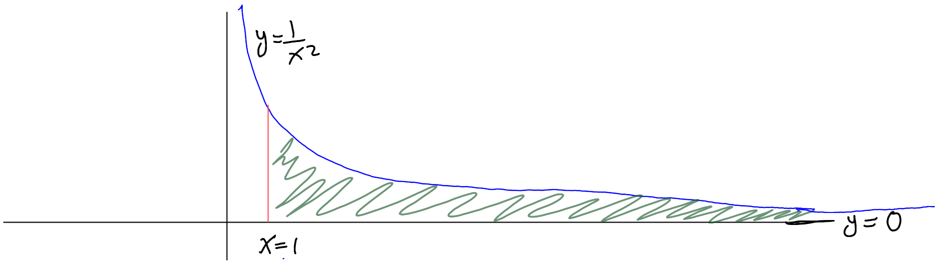
Section 7.8: Improper Integrals

Objective: In this lesson, you learn

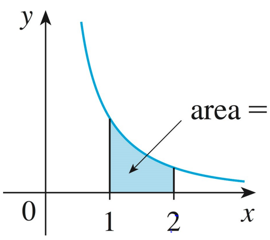
- ☐ how to evaluate a definite integral where the interval is infinite or where f has an infinite discontinuity on $[a, b]$.

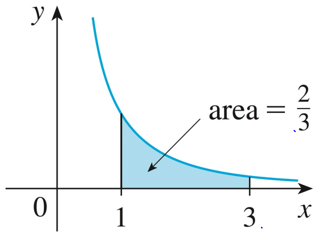
Problem: Evaluate the area of the region bounded by the curves

$$y = \frac{1}{x^2}, \; y = 0, \; x = 1.$$



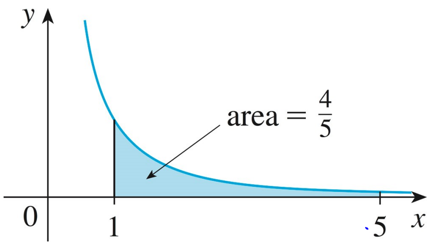
$\int_1^2 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^2 = \left(\frac{-1}{2}\right) - \left(\frac{-1}{1}\right) = 1 - \frac{1}{2} = 1/2$

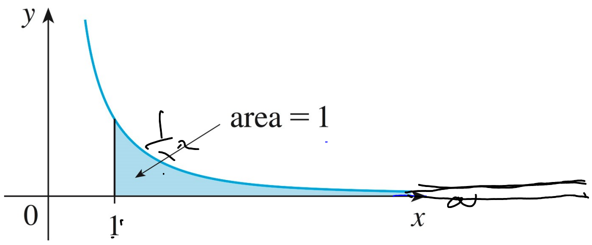




$\int_1^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^3 = \frac{-1}{3} - \left(\frac{-1}{1}\right) = 1 - 1/3 = 2/3$

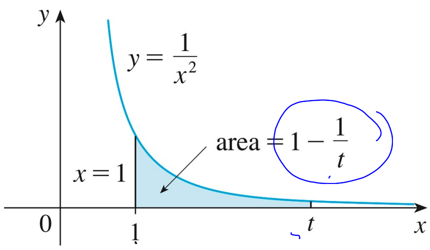
$\int_1^5 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^5 = \frac{-1}{5} - \left(\frac{-1}{1}\right) = 1 - \frac{1}{5} = \frac{4}{5}$





$\lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - \lim_{t\to\infty}\frac{1}{t} = 1 - 0 = 1$

$\int_1^t \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^t = \frac{-1}{t} - \left(\frac{-1}{1}\right) = 1 - \frac{1}{t}$



$t = 2 \Rightarrow 1 - \frac{1}{2} = 1/2$

$t = 3 \Rightarrow 1 - \frac{1}{3} = 2/3$

$t = 5 \Rightarrow 1 - \frac{1}{5} = 4/5$

$\boxed{\lim_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0}$

$\lim_{x\to\infty} e^{-x} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0.$

$\boxed{\lim_{x\to\infty} e^{x} = e^{\infty} = \infty}$

$\lim_{x\to-\infty} e^{x} = e^{-\infty} = 0$

$\lim_{x\to-\infty} e^{-x} = e^{\infty} = \infty$

$\lim_{x\to\infty} \ln x = \ln\infty = \infty$

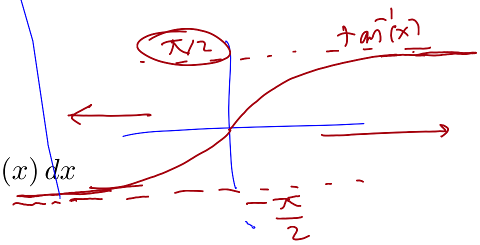
$\lim_{x\to 0^+} \ln x = \ln 0^+ = -\infty$

$\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$

$\lim_{x\to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

I. Type 1: Infinite Intervals.

First, we define the integral of f over an infinite interval as the limit of integrals over finite intervals as follows:



a. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists.

b. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

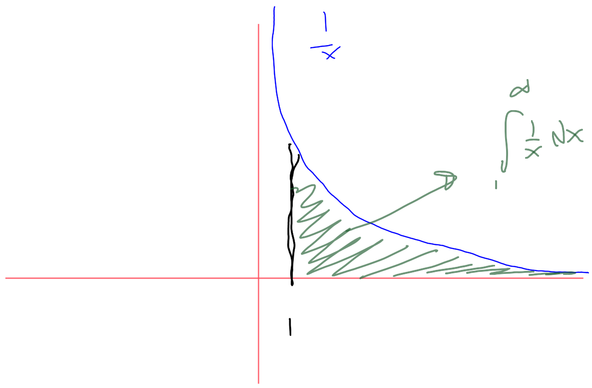
provided this limit exists.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding **limit exists** and **divergent** if the limit **does not exist**. If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Note that any of the improper integrals in the definition above can be interpreted as an **area** if f is a **positive function**.

Example 1: Evaluate $\int_1^\infty \frac{dx}{x}$ if possible.



Consider

$\int_1^t \frac{1}{x}\,dx = \ln x \Big|_1^t$

$= \ln t - \cancel{\ln 1}^{\,0} = \ln t.$

$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$ divergent.

$x = 1$

$x=$	$\frac{1}{x}$	$\frac{1}{x^2}$
1	1	1
10	0.1	0.01
100	0.01	0.0001

Example 2: Evaluate $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ if possible.

consider

$$\int_1^t \frac{dx}{\sqrt{x}} = \int_1^t x^{-1/2}\, dx = \left.\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right|_1^t = 2\, x^{1/2}\Big|_1^t$$

$$= 2(\sqrt{t} - 1)$$

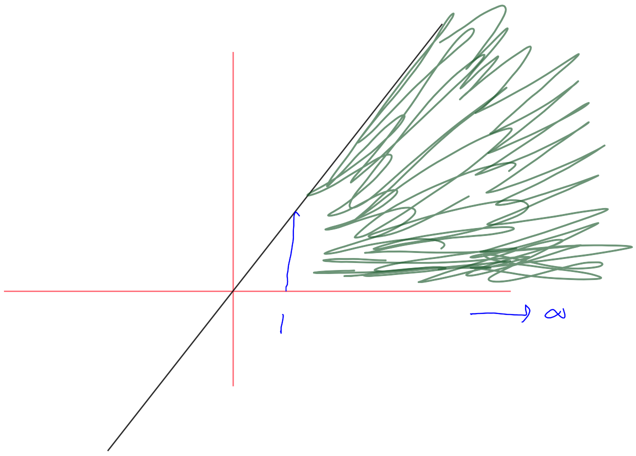
Now,

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} 2(\sqrt{t}-1)$$

$$= \boxed{2(\sqrt{\infty} - 1)} = \infty$$

divergent.

Example 3: Evaluate $\int_1^{\infty} x\, dx$ if possible.



Consider

$$\int_1^t x\, dx = \left.\frac{x^2}{2}\right|_1^t = \frac{t^2}{2} - \frac{1}{2}$$

$$\int_1^{\infty} x\, dx = \lim_{t\to\infty} \int_1^t x\, dx = \lim_{t\to\infty} \frac{1}{2}(t^2-1)$$

$$= \boxed{\frac{1}{2}(\infty^2 - 1)} = \infty$$

divergent.

$\frac{1}{x^{-1}} = x$

$\boxed{\text{p- test}}$

Example 4: Determine for which values of p the integral is convergent or divergent.

$$\int_1^\infty \frac{dx}{x^p}.$$

consider

$$\int_1^t \frac{dx}{x^p} = \int_1^t x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Bigg|_1^t \quad , \quad p \neq 1$$

$$= \frac{1}{-p+1}\cdot\left(t^{-p+1} - 1\right)$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p} =$$

$$= \lim_{t\to\infty} \frac{1}{-p+1}\left(t^{-p+1} - 1\right)$$

$$= \begin{cases} \text{convergent} & p>1 \\ \text{divergent} & p=1 \\ \text{divergent} & p<1 \end{cases}$$

$p=2$

$$\lim_{t\to\infty} \frac{1}{-2+1}\left(t^{-1}-1\right)$$

$$= \lim_{t\to\infty} -\left(\frac{1}{t}-1\right)$$

$$= -\left(\cancel{\tfrac{1}{\infty}}^{\,0}-1\right)$$

$$= 1$$

$-p+1$

if $p>1$

$-p+1 =$ negative

$p<1$

$-p+1 =$ positive

$p=1$

$\Rightarrow$

$$\int_1^\infty \frac{1}{x}\,dx = \ln x\Big|_1^\infty$$

$$= \lim_{t\to\infty} \ln x\Big|_1^t$$

$$= \lim_{t\to\infty} \ln t - \cancel{\ln(1)}^{\,0}$$

$$= \ln\infty = \infty$$

$p=-2$

$$\lim_{t\to\infty} \frac{1}{-(-2)+1}\cdot\left(t^{-(-2)+1}-1\right)$$

$$\lim_{t\to\infty} \frac{1}{3}\left(t^3-1\right)$$

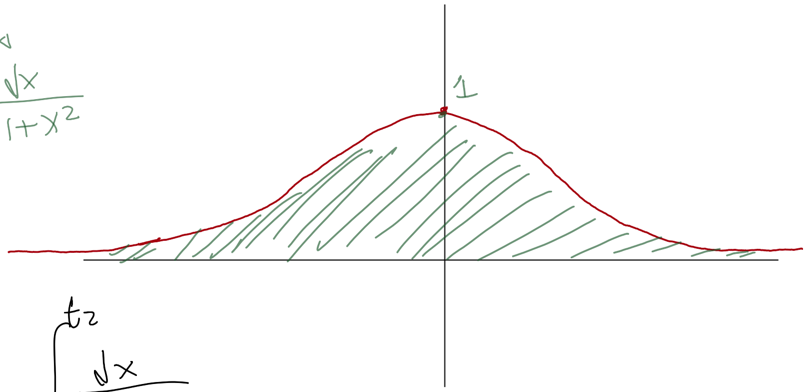
$$= \frac{1}{3}\left(\infty^3-1\right) = \infty$$

$\frac{1}{\pi}\left(\frac{1}{1+x^2}\right)$ Cauchy distribution.

Example 5: Determine the convergence or divergence of

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$



consider

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x)$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{t_1\to-\infty} \int_{t_1}^{0} \frac{dx}{1+x^2} + \lim_{t_2\to\infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1\to-\infty} \tan^{-1}(x)\Big|_{t_1}^{0} + \lim_{t_2\to\infty} \tan^{-1}(x)\Big|_{0}^{t_2}$$

$$= \lim_{t_1\to-\infty}\left(0-\tan^{-1}(t_1)\right) + \lim_{t_2\to\infty}\left(\tan^{-1}(t_2)-0\right)$$

$$= \lim_{t_1\to-\infty} -\tan^{-1}(t_1) + \lim_{t_2\to\infty} \tan^{-1}(t_2) = -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} = \pi.$$

Example 6: Determine the convergence or divergence of

$$\int_0^{\infty} (1-x)e^{-x}\, dx$$

consider

$$\int (1-x)e^{-x}\, dx = -(1-x)e^{-x} + e^{-x} + C$$

u	dv
$1-x$	e^{-x}
-1	$-e^{-x}$
0	e^{-x}

$$\int_0^t (1-x)e^{-x}\,dx = -(1-x)e^{-x} + e^{-x}\Big|_0^t$$

$$= \left(-(1-t)e^{-t} + e^{-t}\right) - \left(-(1-0)\cancel{e^{-0}} + \cancel{e^{-0}}\right)$$

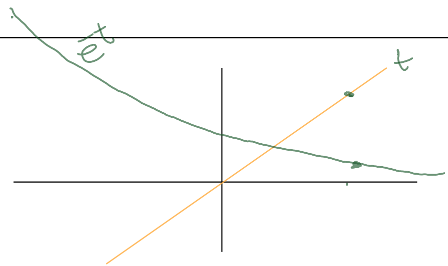
$$= \left((-1+t)e^{-t} + e^{-t}\right) - \cancel{(-(1)+1)}\ 0$$

$$= \left(-\cancel{e^{-t}} + te^{-t} + \cancel{e^{-t}}\right) = te^{-t}$$

$$\int_0^{\infty} (1-x)e^{-x}\,dx = \lim_{t\to\infty} \int_0^t (1-x)e^{-x}\,dx = \lim_{t\to\infty} te^{-t} = (\infty)(e^{-\infty}) \Longrightarrow 0$$

Convergent.

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{e^{\infty}} = 0$$



Example 7: Determine the convergence or divergence of

$$\int_{-\infty}^{0} \frac{dx}{\sqrt{1-x}}$$

$$\int \frac{dx}{\sqrt{1-x}} = \int \frac{-du}{\sqrt{u}} = -\int u^{-1/2}\, du$$

$u = 1-x \longrightarrow du = -dx$

$$= -\frac{u^{1/2}}{1/2} = -2\sqrt{u}$$

$$= -2\sqrt{1-x} + C.$$

$$\int_t^0 \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x}\,\Big|_t^0 = -2\left(\sqrt{1-0} - \sqrt{1-t}\right)$$

$$= -2\left(1-\sqrt{1-t}\right)$$

$$\int_{-\infty}^{0} \frac{dx}{\sqrt{1-x}} = \lim_{t \to -\infty} \int_t^0 \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{t \to -\infty} -2\left(1-\sqrt{1-t}\right)$$

$$= \boxed{-2\left(1-\sqrt{1-(-\infty)}\right)}$$

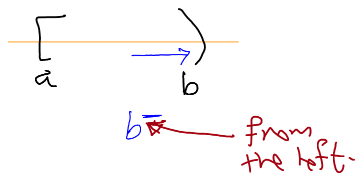
$= \infty$ divergent.

II. Type 2: Discontinuous Integrands.

We define the integral of f over an open interval or a half-open interval as follows.

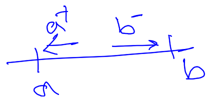
a. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

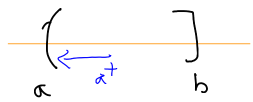


if this limit exists (as a finite number).

b. If f is continuous on $(a, b]$ and is discontinuous at a, then



$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$



if this limit exists (as a finite number).

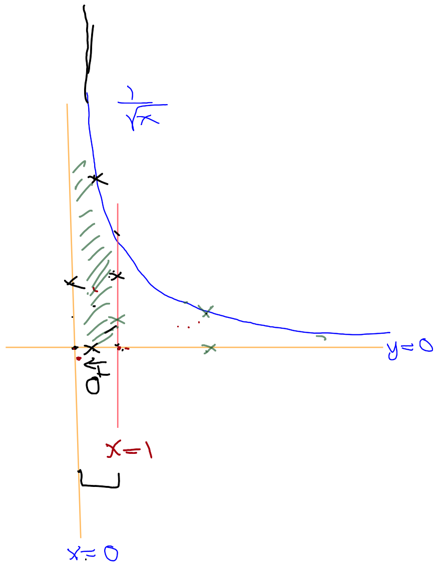
The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

c. If f has a discontinuity, at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 8: Evaluate the area of the region bounded by the curves

$$y = \frac{1}{\sqrt{x}},\ y = 0,\ x = 0, \text{ and } x = 1$$



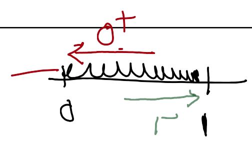
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x} + C.$$

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\,\Big|_t^1 = 2(\sqrt{1}-\sqrt{t}) = 2(1-\sqrt{t})$$

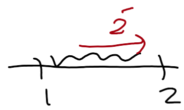
$$\lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}} = \lim_{t\to 0^+} 2(1-\sqrt{t}) = 2(1-\sqrt{0^+}) = 2(1-0) = 2.$$

convergent.



Example 9: Determine whether $\int_1^2 \frac{dx}{(x-2)^2}$ is convergent or divergent.

$$\int_1^2 \frac{dx}{(x-2)^2}\,dx = \lim_{t\to 2^-}\int_1^t \frac{1}{(x-2)^2}\,dx$$



$$\int \frac{dx}{(x-2)^2} \underset{du=dx}{\overset{u=x-2}{=\!=}} \int \frac{1}{u^2}\,dx = \int u^{-2}\,dx = -u^{-1} = \frac{-1}{x-2}+C$$

$$\int_1^t \frac{dx}{(x-2)^2} = \frac{-1}{x-2}\Big|_1^t = \left(\frac{-1}{t-2}\right)-\left(\frac{-1}{-1-2}\right) = \frac{-1}{t-2}-\frac{1}{3}$$

$$\lim_{t\to 2^-}\int_1^t \frac{1}{(x-2)^2}\,dx = \lim_{t\to 2^-}\left(\frac{-1}{t-2}-\frac{1}{3}\right) = \infty$$

$$\frac{-1}{2^- - 2} = \frac{-1}{0^-} = \infty$$

divergent.

Example 10: Determine whether $\int_0^2 \frac{dx}{(2x-1)^{2/3}}$ is convergent or divergent.

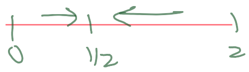
$2x-1=0$
$x=1/2$

$$\boxed{\begin{array}{l} u=2x-1 \\ du=2\,dx \\ dx=\frac{du}{2} \end{array}}$$

$$\int \frac{dx}{(2x-1)^{2/3}} = \frac{1}{2}\int \frac{1}{u^{2/3}}\,du = \frac{1}{2}\cdot\int u^{-2/3}\,du = \frac{1}{2}\,\frac{u^{1/3}}{1/3} + C$$

$$= \frac{3}{2}(2x-1)^{1/3} + C.$$

$$\int_0^2 \frac{dx}{(2x-1)^{2/3}} = \int_0^{1/2} \frac{dx}{(2x-1)^{2/3}} + \int_{1/2}^2 \frac{dx}{(2x-1)^{2/3}}$$



$$= \lim_{t_1\to\frac{1}{2}^-} \int_0^{t_1} \frac{dx}{(2x-1)^{2/3}} + \lim_{t_2\to\frac{1}{2}^+} \int_{t_2}^2 \frac{dx}{(2x-1)^{2/3}}$$

$$= \frac{3}{2}\lim_{t_1\to\frac{1}{2}^-} (2x-1)^{1/3}\Big|_0^{t_1} + \frac{3}{2}\lim_{t_2\to\frac{1}{2}^+} (2x-1)^{1/3}\Big|_{t_2}^2$$

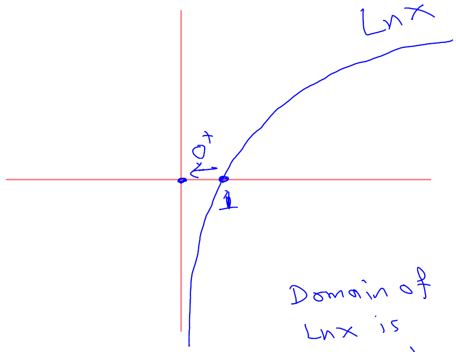
$$= \frac{3}{2}\lim_{t_1\to\frac{1}{2}^-}\left[\cancelto{0^-}{(2t_1-1)^{1/3}} - (-1)^{1/3}\right] + \frac{3}{2}\lim_{t_2\to\frac{1}{2}^+}\left[(3)^{1/3} - \cancelto{0^+}{(2t_2-1)^{1/3}}\right]$$

$$= \frac{3}{2}(0+1) + \frac{3}{2}\left((3)^{\frac{1}{3}} - 0\right)$$

$$= \frac{3}{2}\left(+1 + \sqrt[3]{3}\right)$$

Example 11: Determine whether $\int_0^1 \ln x \, dx$ is convergent or divergent.

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$



Domain of $\ln x$ is $(0, \infty)$

$$\int \ln x \, dx \overset{\text{parts}}{=} x \ln x - x + C$$

$$\int_t^1 \ln x \, dx = x \cdot \ln x - x \Big|_t^1 = (1 \cancel{\ln 1}^{0} - 1) - (t \ln t - t)$$

$$= -1 - t \ln t + t$$

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} (-1 - t \cancel{\ln t}^{0} + \cancel{t}^{0})$$

$\lim_{x \to a} C = C$ if C is a constant

$$= \lim_{t \to 0^+} -1 - \cancel{\lim_{t \to 0^+} t \ln t}^{0} + \lim_{t \to 0^+} t$$

$$= -1 - 0 + 0 = -1$$

$$\lim_{t \to 0^+} t \ln t = 0^+ \ln 0^+ = 0^+ * (-\infty)$$

$$\frac{1}{\frac{1}{t}} = t$$

$$\lim_{t \to 0^+} t \ln t \overset{\frac{\infty}{\infty} \text{ or } \frac{0}{0}}{=} \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} \cdot \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t \to 0^+} \frac{\frac{1}{\cancel{t}}}{\frac{-1}{\cancel{t^2}}}$$

$$= \lim_{t \to 0^+} -t = 0$$

$$\lim_{t \to 0^+} t \ln t = 0$$

III. A Comparison Test for Improper Integrals.

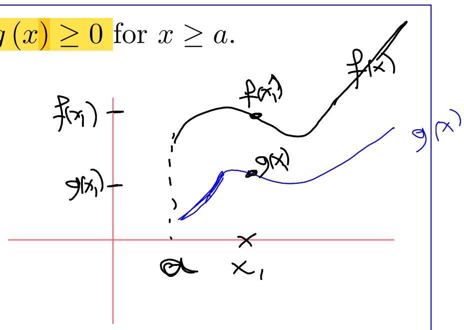
Sometimes it is difficult to evaluate the exact value of an improper integral. However, using the following theorem, its convergence may be determined.

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a. If $\int_a^\infty f(x)\,dx$ is convergent, then
$$\int_a^\infty g(x)\,dx$$
is convergent.

b. If $\int_a^\infty g(x)\,dx$ is divergent, then
$$\int_a^\infty f(x)\,dx$$
is divergent.



Remark: This theorem is plausible if the integrals are considered as the areas under the curves.

- If the area under the top curve, $y = f(x)$, is finite, then so is the area under the bottom curve $y = g(x)$.
- If the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$.

Example 12: Determine whether $\int_4^\infty \frac{dx}{\ln x - 1}$ is convergent or divergent.

$\ln x < x$ for $x \geq 4$

$\ln x - 1 < x - 1$ for $x \geq 4$

$\underbrace{\frac{1}{\ln x - 1}}_{f(x)} > \underbrace{\frac{1}{x-1}}_{g(x)}$ for $x \geq 4$

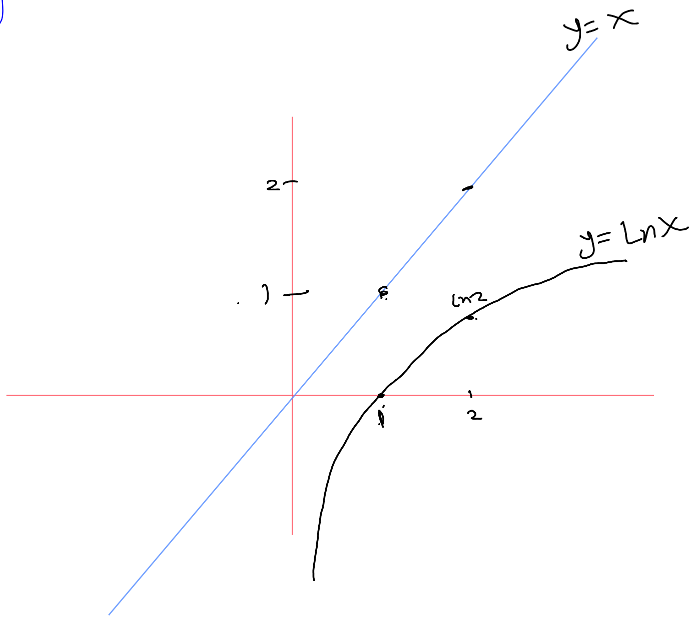
$$\int_4^\infty \frac{1}{\ln x - 1}\,dx > \int_4^\infty \frac{1}{x-1}\,dx$$

is divergent

This also divergent

$$\int_4^\infty \frac{1}{x-1}\,dx = \lim_{t\to\infty} \ln(x-1)\Big|_4^t = \lim_{t\to\infty}\left(\ln(t-1) - \ln 3\right) = \infty$$

$\ln \infty = \infty$



Example 13: Determine whether $\int_1^\infty e^{-x^2/2}\,dx$ is convergent or divergent.

$$x \le x^2 \qquad \text{for } x \geq 1$$

$$-x \geq -x^2$$

$$\frac{-x}{2} \geq \frac{-x^2}{2}$$

$$\int_1^\infty e^{-x/2}\,dx \geq \int_1^\infty e^{-x^2/2}\,dx$$

$$\int_1^\infty e^{-x^2/2}\,dx \le \int_1^\infty e^{-x/2}\,dx \quad \text{is convergent}$$

This is also convergent.

$$\int_1^\infty e^{-x/2}\,dx = \lim_{t\to\infty} -2e^{-x/2}\Big|_1^t$$

$$= \lim_{t\to\infty} -2\left(e^{-t/2} - e^{-1/2}\right)$$

$$= -2\left(\cancel{e^{-\infty/2}}^{\,0} - e^{-1/2}\right)$$

$$= 2e^{-1/2}$$

Example 12: Determine whether $\int_1^\infty \frac{x^2}{\sqrt{x^8+2}}\,dx$ is convergent or divergent.

$(x^4)^2 = x^8$

$(x^2)^4 = x^8$

$x^8 + 2 > x^8$

$\sqrt{x^8+2} > \sqrt{x^8} = x^4$

$\frac{1}{\sqrt{x^8+2}} < \frac{1}{x^4}$

$\frac{x^2}{\sqrt{x^8+2}} < \frac{x^2}{x^4} = \frac{1}{x^2}$

$\int_1^\infty \frac{x^2}{\sqrt{x^8+2}}\,dx < \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \frac{-1}{x}\Big|_1^t = \lim_{t\to\infty} \cancel{\frac{-1}{t}}^{\,0} + 1 = 1$

$\rightarrow$ convergent.